Int. J. Solids Structures, 1974, Vol. 10, pp. 1445–1458. Pergamon Press. Printed in Gt. Britain.

ANALYSIS OF DIFFUSE PLASTIC STABILITY IN TUBES AND SHEETS

E. DUNCOMBE

Bettis Atomic Power Laboratory, West Mifflin, Pennsylvania 15122, U.S.A.

(*Received* 1 *September* 1973; *revised* 29 *March* 1974)

Abstract—The diffuse plastic instability in tubes and sheets under biaxial stress conditions is examined by the use of perturbation methods. Very general constitutive relationships for material properties are used. This requires the inclusion of first order changes in the strain directions inside the patch and also treatment of the material anisotropy and strain rate sensitivity in addition to strain hardening. The inclusion of variations in strain direction is found to alter the form of the characteristic equation for stability from first order to second order but both roots are real for all cases investigated. The value of strain hardening at which the largest root becomes significantly positive is almost the same as that reached when changes in strain directors are ignored. However the strain hardening at which this root becomes formally zero can be very different. The former condition is considered to be of more practical importance than the latter. By this test the stability increases rapidly above a strain rate sensitivity of about 0·1.

1. INTRODUCTION

Plastic instability in tubes and sheets may be defined as a condition whereby a small localized region, which is thinner than the surrounding material of nominal size, tends to progressively decrease in thickness relative to its surroundings. This condition is similar to the corresponding necking instability of bars in a uniaxial stress condition. However in this case the two-dimensional stress field must be taken into account. The presence of significant stresses in the thickness direction also influences stability, but this situation is not appropriate here. Likewise, bending moments and shear stresses in the sheet are assumed small enough to have only minor effects.

Although the locally thinned region may take on a variety of geometrical shapes it is convenient to consider specific modes and to assume that absolute stability occurs if all modes are stable. For instance, the locally thinned region may take this form of a groove which opens up into a cracklike defect. Alternatively it may have finite dimensions in both directions, taking on the form of a diffuse patch. Stability against progressive opening of grooves has been examined in a previous paper[1]. Also analysis of conditions under which localized thinned patches can form in tubes and sheets under biaxial stress conditions has been carried out by a number of previous workers[2–4]. The stability conditions covered here relate to the latter mode with certain additional effects added. The added effects are (1) the influence of strain rate sensitivity, (2) anisotropic material properties, and (3) the fact that strain rate directors themselves vary with the altered stress condition inside the patch.

2. MATERIAL PROPERTIES

In addition to the biaxial stress condition other major factors entering into stability are the material constitutive properties. These can be expressed in terms of differentials as follows

$$d\sigma_g/\sigma_g = m\, d\dot{\varepsilon}_g/\dot{\varepsilon}_g + n\, d\varepsilon_g/\varepsilon_g \tag{1}$$

where σ_g is the generalized stress, a function of the principal stresses, and ε_g is the generalized strain in the material, being a function of the three principal strains. These will be defined later. A dot over a variable indicates the time rate of change. Symbols m and n for present purposes can be considered as numbers which are characteristic of the material in the neighborhood of σ_g, $\dot{\varepsilon}_g$, ε_g but not necessarily outside this neighborhood. The parameter m is usually called the strain rate sensitivity and the parameter n the strain hardening exponent. It is often convenient to replace n/ε_g in equation (1) by a single parameter γ, called the strain hardening parameter. The generalized stress σ_g in the constitutive relation in equation (1) is defined in a manner similar to Hill's definition[5] as

$$\sigma_g = \left[\frac{R(\sigma_h - \sigma_t)^2 + RP(\sigma_t - \sigma_l)^2 + P(\sigma_l - \sigma_h)^2}{RP + P}\right]^{1/2} \tag{2}$$

where σ_h, σ_t and σ_l are the principal stresses in thickness, width and length directions and the anisotropic parameters R and P determine the material strength in each direction. The generalized strain rate $\dot{\varepsilon}_g$ is defined by the work law:

$$\text{Rate of plastic work} = \sigma_g \dot{\varepsilon}_g = \sigma_h \dot{\varepsilon}_h + \sigma_t \dot{\varepsilon}_t + \sigma_l \dot{\varepsilon}_l. \tag{3}$$

Assuming orthogonality of strain rate components to the locus of constant σ_g in stress space[6] it can be shown that

$$\dot{\varepsilon}_h = \frac{\partial \sigma_g}{\partial \sigma_h} \dot{\varepsilon}_g = D_h \dot{\varepsilon}_g \tag{4}$$

$$\dot{\varepsilon}_t = \frac{\partial \sigma_g}{\partial \sigma_t} \dot{\varepsilon}_g = D_t \dot{\varepsilon}_g \tag{5}$$

$$\dot{\varepsilon}_l = \frac{\partial \sigma_g}{\partial \sigma_l} \dot{\varepsilon}_g = D_l \dot{\varepsilon}_g \tag{6}$$

which defines the directors D_h, D_t and D_l. If σ_h is small enough to be neglected the expression for the directors D_h, D_t and D_l obtained by differentiating equation (2), are as follows,

$$D_h = \frac{-(R + P\alpha)}{RP + P} \beta \tag{7}$$

$$D_t = \frac{RP(1 - \alpha) + R}{RP + P} \beta \tag{8}$$

$$D_l = \frac{-RP(1 - \alpha) + P\alpha}{RP + P} \beta \tag{9}$$

where

$$\alpha = \sigma_l/\sigma_t \tag{10}$$

$$\beta = \sigma_t/\sigma_g \tag{11}$$

$$= [(RP + P)^{1/2}/(R + RP(\alpha - 1)^2 + P\alpha^2)^{1/2}](\sigma_t/|\sigma_t|). \tag{12}$$

It is assumed here that $\dot{\varepsilon}_h$, $\dot{\varepsilon}_l$ and $\dot{\varepsilon}_t$ are the total strain rates, i.e. the additional elastic contribution to the strain rates in equations (4,)–(6) are assumed to be negligible compared with plastic strains.

At this stage it is convenient to define for later use the second derivatives of σ_g, as follows:

$$D_{ll} = (\partial^2\sigma_g/\partial\sigma_l^2)\sigma_g = 1 - D_l^2 \tag{13}$$

$$D_{tt} = (\partial^2\sigma_g/\partial\sigma_t^2)\sigma_g = \frac{RP + R}{RP + P} - D_t^2 \tag{14}$$

$$D_{lt} = D_{tl} = (\partial^2\sigma_g/\partial\sigma_l\partial\sigma_t)\sigma_g = \frac{-RP}{RP + P} - D_l D_t. \tag{15}$$

It may be noted that the constitutive relation in equation (1), in the absence of further information, should for generality be written as a vector relationship between the principal stresses σ_h, σ_t and σ_l and the principal strain rates $\dot{\varepsilon}_h$, $\dot{\varepsilon}_t$ and $\dot{\varepsilon}_l$. The alternative procedure of using a single equation via a stress function σ_g defined by equation (2) may be regarded as merely a two parameter (R and P) fitting process, which is not exact but which is usually sufficiently accurate[6].

3. ANALYSIS METHODS

Necessary prerequisites to the analysis of stability are (1) an exact definition of stability, (2) a selection of the variable being examined for stability, and (3) definition of the boundary conditions relating stress and strain conditions outside the patch and those inside. These will now be considered.

Definition of stability

The criterion of Lyapunov, often described as his "first method"[7], will be used here. Briefly, the equilibrium points of the system are first determined and then the dynamic equations are linearized into homogeneous equations for small departures from these equilibrium points. The eigenvalues (latent roots) of these linearized equations are then determined. If any of the eigenvalues are positive the system is unstable. The equations themselves are strictly applicable only to local regions around the equilibrium point over which the linearization is valid, and thus this definition is sometimes referred to as "stability in the small" rather than stability "in the large" which would cover the global situation.

The variable being examined

The specific variable chosen here is the difference $\delta\varepsilon_g$ between generalized strain in the necked region and that in the surrounding unnecked region (Fig. 1). The symbol δ is always used hereafter to denote a difference between the value of some parameter inside the necked region and the value in the material outside this region at the same instant of time. Variables other than $\delta\dot{\varepsilon}_g$ could be chosen. For instance, it would be feasible to choose the difference in thickness δh. Alternatively the difference in thickness strain could be chosen, i.e. $\delta\varepsilon_h$. The three variables are related as follows:

$$\delta\dot{\varepsilon}_h = \delta(D_h\dot{\varepsilon}_g) = D_h\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\sum_i \frac{\partial D_h}{\partial\sigma_i}\delta\sigma_i \tag{16}$$

$$\delta\dot{\varepsilon}_h = \delta\left(\frac{\dot{h}}{h}\right) = \frac{1}{h}\,\delta\dot{h} - \frac{\dot{h}}{h^2}\,\delta h = \frac{1}{h}\,\delta\dot{h} - \dot{\varepsilon}_h\delta\varepsilon_h. \tag{17}$$

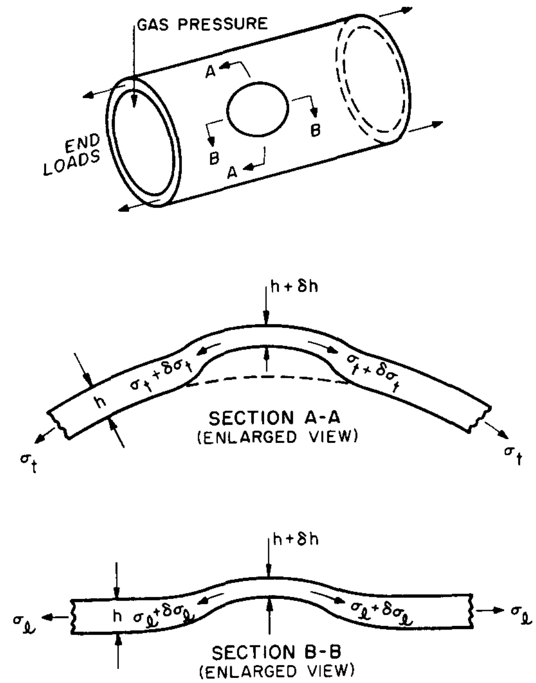

Fig. 1. Locally thinned region of a tube subjected to internal gas pressure and end loads.

Conditions for the stability of these variables would be expected to be similar. They are not identical, since the new variables, $\delta\varepsilon_h$ or δh, involve integrals of the variable $\delta\dot{\varepsilon}_g$, which are formally infinite when the latent roots determining $\delta\dot{\varepsilon}_g$ are zero. For instance, if $\delta\dot{\varepsilon}_g = K \exp(\lambda t)$, then $\lambda = 0$ when $[\mathrm{d}(\delta\dot{\varepsilon}_g)/\mathrm{d}t]_{t=0} = 0$. However, $\lambda < 0$ when $(\mathrm{d}/\mathrm{d}t)[\delta\dot{\varepsilon}_g + \int \delta\dot{\varepsilon}_g\,\mathrm{d}t]_{t=0} = 0$.

Boundary conditions

The boundary conditions assumed here are illustrated in Fig. 1, which shows a tube under internal pressure and end loads. The basic assumptions are that the principal stresses outside the thinned region are amplified inside the region by the ratio of thicknesses in these locations.

The possibility of localized thinning is most evident in the case of drawing processes[8, 9] in which case the locally thinned patches would be expected to become more highly stressed as they become thinner. In the case of flat plates localized thinning would lead to unloading and in such cases other instability mechanisms such as the growth of groove-like defects may be expected to dominate[1]. Internally pressurized tubes could develop instabilities in either mode depending on dimensions and material properties. Usually in pressurized tube tests the initial instability is diffuse (patch-type) although the final tear may be associated with groove instability.

Adopting the boundary assumptions as stated, the following relationships apply to the membrane stresses σ_l and σ_t

$$\delta\sigma_l/\sigma_l = -\delta A_l/A_l = \delta\varepsilon_l \quad (18)$$

$$\delta\sigma_t/\sigma_t = -\delta A_t/A_t = \delta\varepsilon_t \quad (19)$$

where δA_i is the difference in area of cross section over which the applied force in direction i acts. The second equality in each of equations (18) and (19) arises from the assumption of constant volume.

4. CHARACTERISTIC EQUATION FOR STABILITY

It is shown in the Appendix that the constitutive equation (1) and the boundary conditions, equations (18 and 19), together with the expression for generalized stress, equation (2), and the strain rate equations (5) and (6) are sufficient to specify a set of homogeneous equations in the variables $\delta\sigma_g$, $\delta\sigma_l$, $\delta\sigma_t$, $\delta\varepsilon_g$, $\delta\varepsilon_l$ and $\delta\varepsilon_t$. This leads to a characteristic equation in $\delta\varepsilon_g$ which is of the form

$$(As^2 + Bs + C)\delta\dot{\varepsilon}_g = 0 \tag{20}$$

where s stands for the derivative d/dt. The coefficients A, B and C are given by

$$A = m/\dot{\varepsilon}_g \tag{21}$$

$$B = \gamma - (m/\dot{\varepsilon}_g)(D_{ll}\alpha + D_{tt})\beta - \alpha\beta D_l^2 - \beta D_t^2 \tag{22}$$

$$C = -\gamma\dot{\varepsilon}_g(D_{tt}\alpha + D_{tt})\beta + \alpha\beta^2\dot{\varepsilon}_g(D_l^2 D_{tt} - 2D_l D_t D_{lt} + D_t^2 D_{ll}). \tag{23}$$

It may be noted that, when the second derivatives D_{ij} are assumed to be zero, equation (20) becomes

$$\frac{m}{\dot{\varepsilon}_g}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\delta\varepsilon_g) + (\gamma - \alpha\beta D_l^2 - \beta D_t^2)\frac{\mathrm{d}}{\mathrm{d}t}(\delta\varepsilon_g) = 0. \tag{24}$$

This equation gives the same condition for absolute stability (zero roots) as Hillier[4] for his special case (ii) with $R = 1$, $P = 1$, $m = 0$, i.e.

$$(\gamma_0 - \alpha\beta D_l^2 - \beta D_t^2) = 0 \tag{25}$$

which specifies the critical value of γ for specified values of stress conditions defined by α and β. The subscript has been added to γ to indicate that γ_0 is the special value of γ which leads to absolute stability if second derivatives of σ_g, i.e. D_{ij}, are assumed to be zero. It may be noted that the roots of equation (24) are linearly dependent on $\dot{\varepsilon}_g/m$. The solution to equation (24) is of the form

$$\delta\dot{\varepsilon}_g = (\delta\,\dot{\varepsilon}_g)_0 \exp[(\gamma - \alpha\beta D_l^2 - \beta D_t^2)(\dot{\varepsilon}_g/m)\,\Delta t] \tag{26}$$

where $(\delta\dot{\varepsilon}_g)_0$ is the initial value of $\delta\dot{\varepsilon}_g$ at time $\Delta t = 0$. The time constant for the growth or diminution of the perturbation $\delta\dot{\varepsilon}_g$ is, not unexpectedly, proportional to the strain rate sensitivity m and inversely proportional to the nominal strain rate $\dot{\varepsilon}_g$.

Turning now to the full solution of equation (20) including terms in D_{ij} the two roots are

$$\lambda = [-B \pm (B^2 - 4AC)^{1/2}]/2A. \tag{27}$$

With $A = m/\dot{\varepsilon}_g$ it is clear that the time constants of increase or decay in $\delta\dot{\varepsilon}_g$ are again linearly related to $\dot{\varepsilon}_g$ and inversely to m. However, since the numerator in equation (27) is also a function of m (though not of $\dot{\varepsilon}_g$) the m-dependence for λ as a whole is not strictly linear, though this will be nearly so when m is small enough.

The positive sign in equation (27) is identified with the most positive root since both the factors m and $\dot{\varepsilon}_g$ of A in equation (21) are positive (m is positive for realistic properties).

The condition for absolute stability becomes

$$(B^2 - 4AC)^{1/2} < B \tag{28}$$

or, with A and B positive,

$$C < 0. \tag{29}$$

If B is negative then stability is unattainable.

5. DISCUSSION OF RESULTS

Roots of the characteristic equation

Values of the normalized root $\lambda/\dot{\varepsilon}_g$ are shown in Figs. 2–7 for a range of values of $\alpha = \sigma_l/\sigma_t$ and strain hardening parameter γ for typical values of R, P and strain rate sensitivity m. Some of these figures show the smaller root in equation (27) in addition to the larger root although the algebraically larger root determines stability. Observations are as follows:

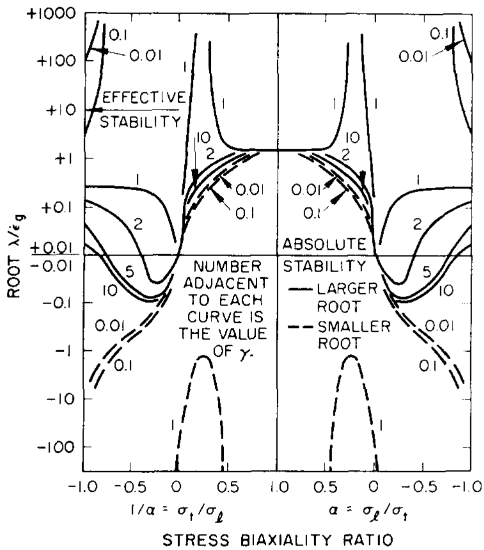

Fig. 2. Roots of equation for neck growth $R = P = 1$, $m = 10^{-5}$.

(1). Stability is always reduced (λ is more positive) as the value of γ is decreased. This follows the direction of the classical stability criterion of Considère[10] for the uniaxial case without m-dependence, being

$$\gamma = \frac{1}{\sigma}\frac{d\sigma}{\partial\varepsilon} < 1. \tag{30}$$

(2). As the strain rate sensitivity m increases, stability is increased in the sense that the roots λ become reduced.

(3). The above conclusions are true for anisotropic properties ($R > 1$, $P > 1$). If $R \neq P$ the roots λ are not the same when σ_l/σ_t is replaced by σ_t/σ_l.

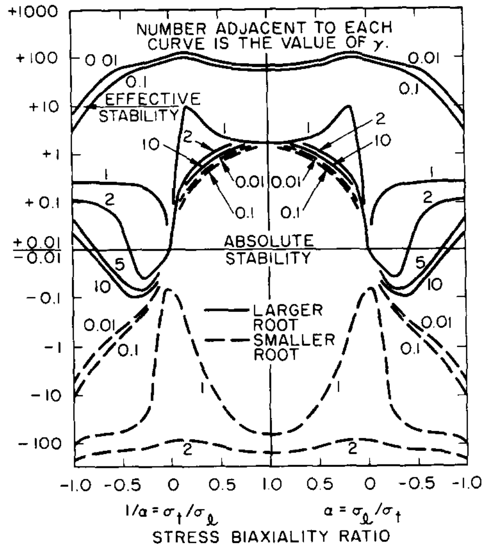

Fig. 3. Roots of equation for neck growth $R = P = 1$, $m = 0{\cdot}01$.

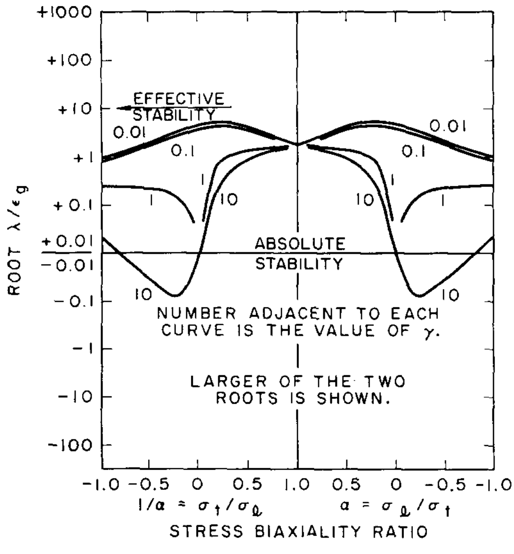

Fig. 4. Roots of equation for neck growth $R = P = 1$, $m = 0{\cdot}25$.

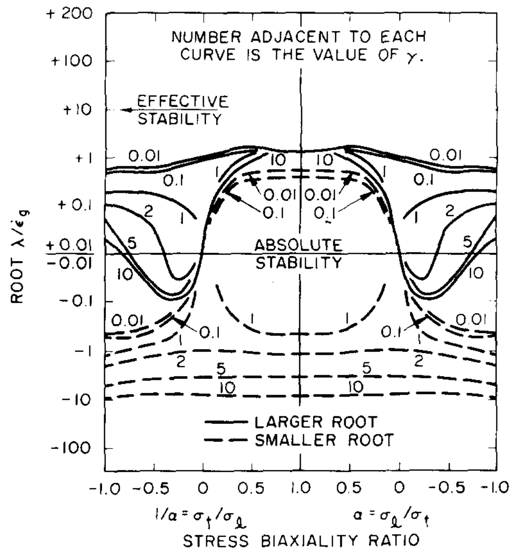

Fig. 5. Roots of equation for neck growth $R = P = 1$, $m = 1$.

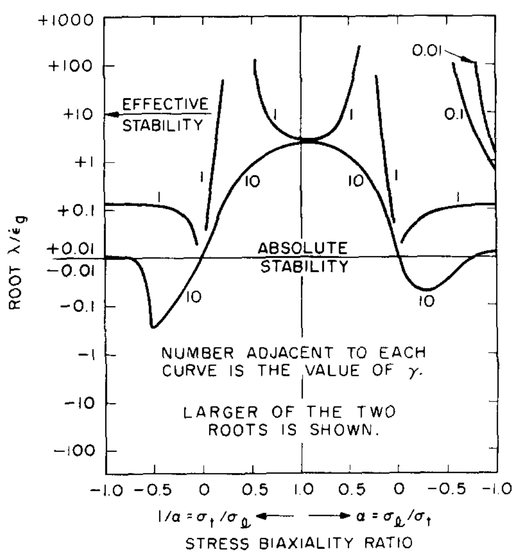

Fig. 6. Roots of equation for neck growth $R = 2$, $P = 4$, $m = 10^{-5}$.

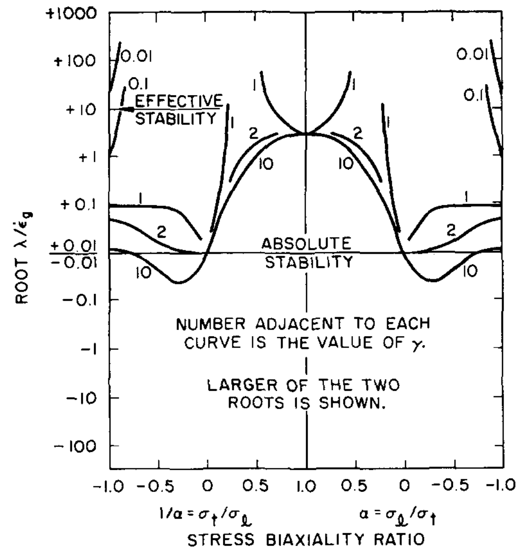

Fig. 7. Roots of equation for neck growth $R = 4$, $P = 4$, $m = 10^{-5}$.

(4). In certain regions of α, absolute stability is never attained (roots are always positive). At first sight this seems paradoxical but this only implies that inhomogeneities will always propagate, though perhaps at negligible rates.

In order to determine conditions for propagation of inhomogeneities at acceptable rates, a solution to equation (20) of the form

$$\delta\dot{\varepsilon}_g \cong K_1 \exp\{(\lambda/\dot{\varepsilon}_g)\dot{\varepsilon}_g \,\Delta t\} \tag{31}$$

is considered, where λ is the largest root in equation (27). An acceptably small value of $\lambda/\dot{\varepsilon}_g$ can be defined as that value which makes the exponential factor in equation (31) acceptably small, say e $= 2{\cdot}718$ for convenience, after some acceptable time Δt. As can be seen from the form of equation (31), if the normalized root $\lambda/\dot{\varepsilon}_g$ is taken, the remaining term in the exponent is $\dot{\varepsilon}_g \,\Delta t$, or $\Delta\varepsilon_g$ within the assumptions of constancy of $\dot{\varepsilon}_g$. Now in practical cases it is sufficient to consider the growth of an inhomogeneity only for some specific maximum strain (here the thinning) of the material outside the inhomogeneity. If this maximum strain is taken as 10 per cent then the allowable value of $\lambda/\dot{\varepsilon}_g$ is determined by

$$2{\cdot}718 = \exp\{(\lambda/\dot{\varepsilon}_g)/10\} \tag{32}$$

or

$$(\lambda/\dot{\varepsilon}_g)_{\text{eff}} = 10 \tag{33}$$

where $(\lambda/\dot{\varepsilon}_g)_{\text{eff}}$ is thus the value of $\lambda/\dot{\varepsilon}_g$ which makes the perturbation strain rate increase by a factor of e $= 2{\cdot}718$ when a nominal strain increment of 10 per cent is reached in the surrounding material. These values of $\lambda/\dot{\varepsilon}_g$ are labelled as the effective stability points in Figs. 2–7. It can be seen that effective stability is achieved, i.e. no "substantial" aggravation of a neck, for values of $\gamma = \gamma_{\text{eff}}$ often considerably lower than those for absolute stability.

Critical values of γ *and* m

Values of the reciprocal strain hardening parameter $\gamma^{-1} = \sigma_g/(\partial\sigma_g/\partial\varepsilon_g)$ for absolute stability (γ_{abs}^{-1}) and for effective stability (γ_{eff}^{-1}) are plotted in Figs. 8–10. These three figures relate to isotropic properties ($R = P = 1$), to anisotropic properties ($R = 2$, $P = 4$) and to anisotropic properties with isotropy between the width and thickness directions ($R = 4$, $P = 4$). Variations with strain rate sensitivity m are also shown in these figures. The value γ_0^{-1} for stability, assuming second derivatives D_{ij} are zero, from equations (24) and (25) is also shown. The same comments are applicable as those made in connection with the plots of λ in Figs. 2–7.

In addition it may be seen that the values of γ_{eff}^{-1} are close to those of γ_0^{-1} for small values of m, say $m \leq 0{\cdot}01$. This is not a coincidence as will now be demonstrated.

If the roots λ of equation (20) are examined for $m \to 0$, in the range $C < 0$ (see Figs. 8–10),

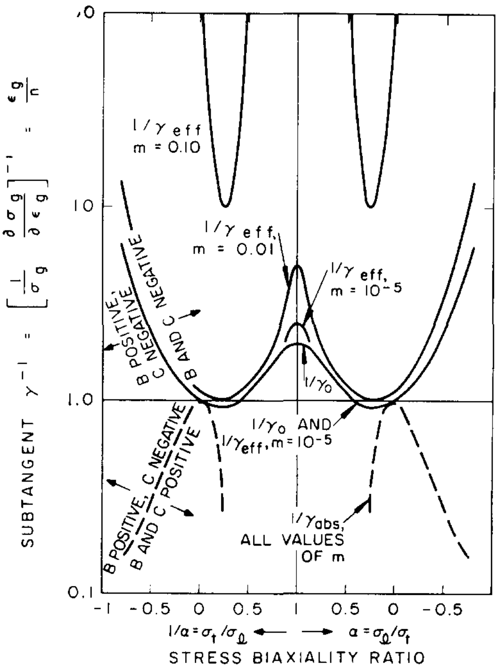

Fig. 8. Absolute and effective stability criteria $R = P = 1$.

assuming as before that A from equation (21) is positive, equation (27) shows that the largest (most positive) values of λ approaches $|C|/B$ if B is positive. As B goes through zero, provided A is finite, λ becomes $(|C|/A)^{1/2}$. However when B becomes negative, with m still vanishingly small, the largest root is now $|B|/A$. Thus a change in sign of B causes the root to go from a low positive value to a high positive value and therefore the effective stability point would be expected to be in the region close to $B = 0$ for small values of m.

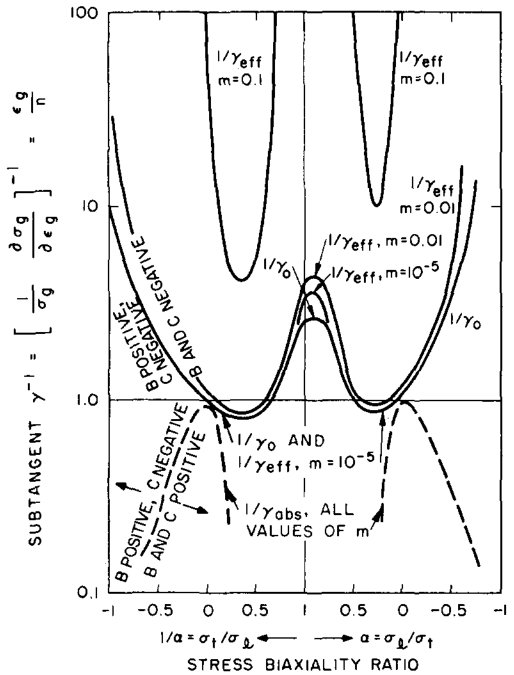

Fig. 9. Absolute and effective stability criteria $R = 2, P = 4$.

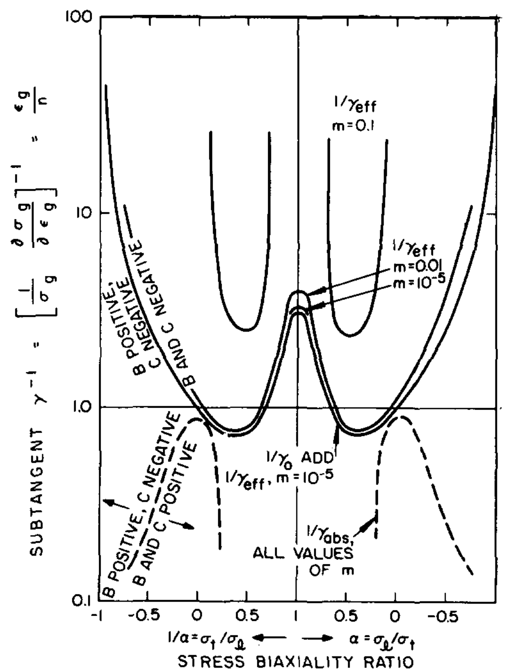

Fig. 10. Absolute and effective stability criteria $R = P = 4$.

6. CONCLUSIONS

(1). Inclusion of first order changes in the strain director D_{ij} inside the patch causes the characteristic equation governing stability to become quadratic whereas exclusion of these changes produces a first order equation.

(2). The roots of the quadratic equation are real in all cases investigated.

(3). For some values of biaxiality ratio σ_l/σ_t the largest root is positive for all values of strain rate sensitivity m and strain hardening parameter γ.

(4). Stability can be redefined as a condition whereby "significant" neck growth does not occur for some prescribed strain (e.g. 10 percent) in the material outside the patch. The effective value of strain hardening parameter, γ_{eff}, for this condition and for $m \to 0$ is very close to the value obtained without inclusion of changes in strain directors. This is shown to be a rational conclusion.

(5). As the strain rate sensitivity m is increased beyond about 0·1 the values of γ_{eff} become very small, indicating high relative stability.

(6). Effects of anisotropy are not large over the ranges investigated.

REFERENCES

1. E. Duncombe, Plastic Instability and Growth of Grooves and Patches in Plates or Tubes, *Int. J. Mech. Sci.* **14**, 325–337 (1972).
2. W. T. Lankford and E. Saibel, Some Problems in Unstable Plastic Flow Under Biaxial Tension, American Institute of Mining and Metallurgical Engineers, *Tech. Publ. No.* 2238, *Class E, Metals Technology* (August 1947).
3. H. W. Swift, Plastic Instability Under Plane Stress, *J. Mech. Phys. Solids* **1**, 1–18 (1952).
4. M. J. Hillier, Tensile Instability of Thin Tubes—I, *Int. J. Mech. Sci.* **7**, 531–538 (1965).
5. R. Hill, *The Mathematical Theory of Plasticity*, pp. 317–320. Clarendon Press Oxford (1950).
6. D. C. Bogue, The Yield Stress and Plastic Strain Theory for Anisotropic Materials, *Oak Ridge Nat. Lab. Rep.* ORNL-TM-1869 (July 1967).
7. A. M. Lyapunov, Probléme Gèneral de la Stabilité de Mouvement, *Annals of Mathematical Studies No.* 17. Princeton University Press (1949).
8. S. P. Keeler and W. A. Backofen, Plastic Instability and Fracture in Sheets Stretched Over Rigid Punches, *Trans. ASM* **56**, 25–48 (1963).
9. M. Azrin and W. A. Backofen, The Deformation and Failure of a Biaxially Stretched Sheet, *Met. Trans.* **1**, 2857–2865 (1970).
10. M. Considère, Memoire sur l'Emploi du Fer et l'Acier Dans les Constructions, *Ann. des Ponts et Chaussées* **9**, Ser. 6, 574 (1885).

Абстракт — Применяя методы возмущения исследуется пластическая неустойчивость диффундирования в трубах и листах, в условиях двухосного напряжения. Используются очень общие конститутивные зависимости для свойств материала. Это требует включения изменений первого порядка в направлениях деформации внутри небольшого участка и, также, рассмотрения анизотропии материала и чувствительности к скорости деформации, в добавлении к деформационному упрочнению. Находится, что включение изменений в направлении деформации изменяет форму характеристического уравнения устойчивости из первого порядка к другому, но оба корни действительны для всех исследованных случаев. Значение деформационного упрочнения, при котором самый большой корень значительно положительный почти тот же самое, как то достигнуто, когда пренебрегаются изменениями в векторах направления деформации. Тем не менее, деформационное упрочнение, при котором этот корень переходит формально в нуль, может быть совсем другое. Первое условие считается быть более практического значения по сравнению с другим. Из этого опыта оказывается, что устойчивость повышается быстро выше чувствительности к скорости деформации порядка приблизительно 0,1.

APPENDIX

Derivation of the characteristic equation for stability

The perturbation variables can be identified as $\delta\sigma_g$, $\delta\sigma_l$, $\delta\sigma_t$, $\delta\varepsilon_l$, $\delta\varepsilon_t$ and $\delta\varepsilon_g$. The time derivatives of variables, such as $\delta\dot{\varepsilon}_g$, are not treated here as new variables, but as $s\,\delta\dot{\varepsilon}_g$ where the multiplier s can be considered as a linear operator, or equivalently, as Laplace transform multiplier. For solution of this system there is first the constitutive equation (1). In addition there are two boundary conditions, equations (18) and (19). Then there is the relation, equation (2), between σ_g, σ_l and σ_t. This can be expressed in perturbation variables as

$$\delta\sigma_g = \frac{\partial\sigma_g}{\partial\sigma_l}\,\delta\sigma_l + \frac{\partial\sigma_g}{\partial\sigma_t}\,\delta\sigma_t$$

$$= D_l\,\delta\sigma_l + D_t\,\delta\sigma_t. \tag{A1}$$

Finally the two relations, equations (5) and (6), exist between $\dot{\varepsilon}_l$ and $\dot{\varepsilon}_g$ and between $\dot{\varepsilon}_t$ and $\dot{\varepsilon}_g$. These can be expressed in perturbation form as

$$\delta\dot{\varepsilon}_l = \delta(D_l\,\dot{\varepsilon}_g) = D_l\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\,\delta D_l$$

$$= D_l\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\left(\frac{\partial D_l}{\delta\sigma_l}\,\delta\sigma_l + \frac{\partial D_l}{\partial\sigma_t}\,\delta\sigma_t\right) \tag{A2}$$

$$= D_l\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\left(D_{ll}\,\frac{\sigma_l}{\sigma_g}\,\frac{\delta\sigma_l}{\sigma_l} + D_{lt}\,\frac{\sigma_t}{\sigma_g}\,\frac{\delta\sigma_t}{\sigma_t}\right)$$

where $D_{ll} = \sigma_g\,\partial D_l/\partial\sigma_l$ and $D_{lt} = \sigma_g\,\partial D_l/\partial\sigma_t$, as given by equations (13) and (15), and similarly

$$\delta\dot{\varepsilon}_t = D_t\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\left(D_{tl}\,\frac{\sigma_l}{\sigma_g}\,\frac{\delta\sigma_l}{\sigma_l} + D_{tt}\,\frac{\sigma_t}{\sigma_g}\,\frac{\delta\sigma_t}{\sigma_t}\right) \tag{A3}$$

where $D_{tt} = \sigma_g\,\partial D_t/\partial\sigma_l = D_{lt}$ and $D_{tt} = \sigma_g\,\partial D_t/\partial\sigma_t$ as given by equations (14) and (15).

The six equations (1, 18, 19, A1, A2 and A3) are thus sufficient to find a solution. The solution equation will be derived in $\delta\varepsilon_g$.

The variable $\delta\sigma_g$ is first eliminated by combining equation (1) and (A1) into

$$\frac{1}{\sigma_g}\,[D_l\,\delta\sigma_l + D_t\,\delta\sigma_t] = m\,\frac{\delta\dot{\varepsilon}_g}{\dot{\varepsilon}_g} + \gamma\,\delta\varepsilon_g \tag{A4}$$

where the new variable γ is used instead of n/ε_g. The system is now reduced to five equations (18, 19, A2–A4) and five variables $\delta\sigma_l$, $\delta\sigma_t$, $\delta\varepsilon_l$, $\delta\varepsilon_t$ and $\delta\varepsilon_g$. The next step is to use equations (18) and (19) to replace $\delta\sigma_l$ and $\delta\sigma_t$ by $\delta\varepsilon_l$ and $\delta\varepsilon_t$. If this is performed in equations (A2) and (A3) these equations become

$$\delta\dot{\varepsilon}_l = D_l\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\left(D_{ll}\,\frac{\sigma_l}{\sigma_g}\,\delta\varepsilon_l + D_{lt}\,\frac{\sigma_t}{\sigma_g}\,\delta\varepsilon_t\right) \tag{A5}$$

$$\delta\dot{\varepsilon}_t = D_t\,\delta\dot{\varepsilon}_g + \dot{\varepsilon}_g\left(D_{tl}\,\frac{\sigma_l}{\sigma_g}\,\delta\varepsilon_l + D_{tt}\,\frac{\sigma_t}{\sigma_g}\,\delta\varepsilon_t\right). \tag{A6}$$

This reduces the variables to $\delta\varepsilon_l$, $\delta\varepsilon_t$ and $\delta\varepsilon_g$ and the equations to (A4)–(A6). It is now necessary to solve equations (A5) and (A6) simultaneously for $\delta\varepsilon_l$ and $\delta\varepsilon_t$ as functions of $\delta\varepsilon_g$ and then put each of these solutions into equation (A4). The simultaneous solution is

$$\delta\varepsilon_l = \frac{(D_l\,\delta\dot{\varepsilon}_g)(s - \dot{\varepsilon}_g\beta D_{tt}) + (\dot{\varepsilon}_g D_{tt}\beta)(D_t\,\delta\dot{\varepsilon}_g)}{(s - \dot{\varepsilon}_g D_{ll}\alpha\beta)(s - \dot{\varepsilon}_g D_{tt}\beta) - \dot{\varepsilon}_g^{\,2} D_{lt}^{\,2}\alpha\beta^2} \tag{A7}$$

$$\delta\varepsilon_t = \frac{(s - \dot{\varepsilon}_g\alpha\beta D_{ll})(D_t\,\delta\dot{\varepsilon}_g) + (D_l\,\delta\dot{\varepsilon}_g)(\dot{\varepsilon}_g D_{lt}\alpha\beta)}{(s - \dot{\varepsilon}_g D_{ll}\alpha\beta)(s - \dot{\varepsilon}_g D_{tt}\beta) - \dot{\varepsilon}_g^{\,2} D_{lt}^{\,2}\alpha\beta^2} \tag{A8}$$

where $\alpha = \sigma_l/\sigma_t$, $\beta = \sigma_t/\sigma_g$ and s is the time operator d/dt.

Finally the above expressions for $\delta\varepsilon_l$ and $\delta\varepsilon_t$ can be inserted into equation (A4), yielding a second order differential equation in only one perturbation variable, $\delta\dot{\varepsilon}_g$. Some simplification occurs in the denominators of equations (A7) and (A8) owing to the identity

$$D_{tt}\,D_{ll} = D_{tl}^{\,2} \tag{A9}$$

which can be shown by using equations (13)–(15). The final equation of motion for $\delta\dot{\varepsilon}_g$ is thus of the form

$$(As^2 + Bs + C)\delta\dot{\varepsilon}_g = 0. \tag{A10}$$

The coefficients A, B and C are given by equations (21)–(23) in the body of the paper.